# Multinomial Logistic Regression

## Introduction

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STA 210

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# Topics

- Introduce multinomial logistic regression
- Interpret model coefficients
- Inference for a coefficient $\beta_{jk}$

# Generalized Linear Models (GLM)

- In practice, there are many different types of response variables including:

  - **Binary**: Win or Lose
  - **Nominal**: Democrat, Republican or Third Party candidate
  - **Ordered**: Movie rating (1 - 5 stars)
  - and others...

- These are all examples of **generalized linear models**, a broader class of models that generalize the multiple linear regression model

- See *Generalized Linear Models: A Unifying Theory* for more details about GLMs

# Binary Response (Logistic)

- Given $P(y_i = 1|x_i) = \hat{\pi}_i \quad \text{and} \quad P(y_i = 0|x_i) = 1 - \hat{\pi}_i$

$$\log\left(\frac{\hat{\pi}_i}{1 - \hat{\pi}_i}\right) = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- We can calculate $\hat{\pi}_i$ by solving the logit equation:

$$\hat{\pi}_i = \frac{\exp\{\hat{\beta}_0 + \hat{\beta}_1 x_i\}}{1 + \exp\{\hat{\beta}_0 + \hat{\beta}_1 x_i\}}$$

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# Binary Response (Logistic)

Suppose we consider $y = 0$ the **baseline category** such that

$$P(y_i = 1|x_i) = \hat{\pi}_{i1} \text{ and } P(y_i = 0|x_i) = \hat{\pi}_{i0}$$

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Then the logistic regression model is

$$\log\left(\frac{\hat{\pi}_{i1}}{1 - \hat{\pi}_{i1}}\right) = \log\left(\frac{\hat{\pi}_{i1}}{\hat{\pi}_{i0}}\right) = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

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**Slope, $\hat{\beta}_1$**: When $x$ increases by one unit, the odds of $y = 1$ versus the baseline $y = 0$ are expected to multiply by a factor of $\exp\{\hat{\beta}_1\}$

**Intercept, $\hat{\beta}_0$**: When $x = 0$, the predicted odds of $y = 1$ versus the baseline $y = 0$ are $\exp\{\hat{\beta}_0\}$

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# Multinomial response variable

- Suppose the response variable $y$ is categorical and can take values $1, 2, \dots, K$ such that $(K > 2)$

- Multinomial Distribution:

$$P(y = 1) = \pi_1, P(y = 2) = \pi_2, \dots, P(y = K) = \pi_K$$

such that $\sum_{k=1}^{K} \pi_k = 1$

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# Multinomial Logistic Regression

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$$\log\left(\frac{\pi_{ik}}{\pi_{i1}}\right) = \beta_{0k} + \beta_{1k}x_i$$

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$$\log\left(\frac{\pi_{ik}}{\pi_{i1}}\right) = \beta_{0k} + \beta_{1k}x_i$$

- In the multinomial logistic model, we have a separate equation for each category of the response relative to the baseline category
  - If the response has $K$ possible categories, there will be $K - 1$ equations as part of the multinomial logistic model

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# Multinomial Logistic Regression

- Suppose we have a response variable $y$ that can take three possible outcomes that are coded as "A", "B", "C"
- Let "A" be the baseline category. Then

$$\log\left(\frac{\pi_{iB}}{\pi_{iA}}\right) = \beta_{0B} + \beta_{1B}x_i$$

$$\log\left(\frac{\pi_{iC}}{\pi_{iA}}\right) = \beta_{0C} + \beta_{1C}x_i$$

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# NHANES Data

- National Health and Nutrition Examination Survey is conducted by the National Center for Health Statistics (NCHS)
- The goal is to *"assess the health and nutritional status of adults and children in the United States"*
- This survey includes an interview and a physical examination

# NHANES Data

- We will use the data from the `NHANES` R package
- Contains 75 variables for the 2009 - 2010 and 2011 - 2012 sample years
- The data in this package is modified for educational purposes and should **not** be used for research
- Original data can be obtained from the [NCHS website] for research purposes
- Type `?NHANES` in console to see list of variables and definitions

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# Health Rating vs. Age & Physical Activity

- **Question:** Can we use a person's age and whether they do regular physical activity to predict their self-reported health rating?
- We will analyze the following variables:
  - **`HealthGen`:** Self-reported rating of participant's health in general. Excellent, Vgood, Good, Fair, or Poor.
  - **`Age`:** Age at time of screening (in years). Participants 80 or older were recorded as 80.
  - **`PhysActive`:** Participant does moderate to vigorous-intensity sports, fitness or recreational activities

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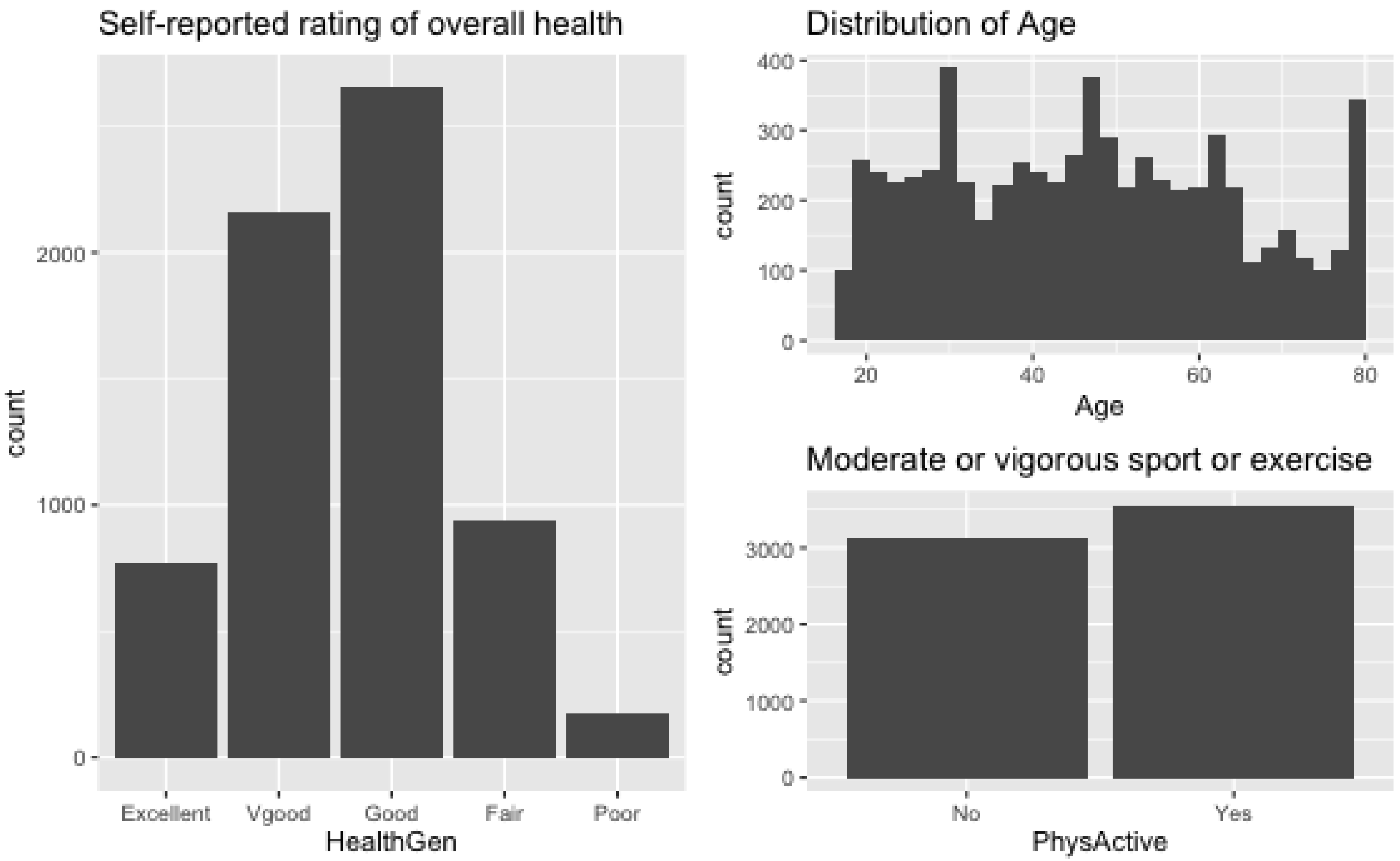
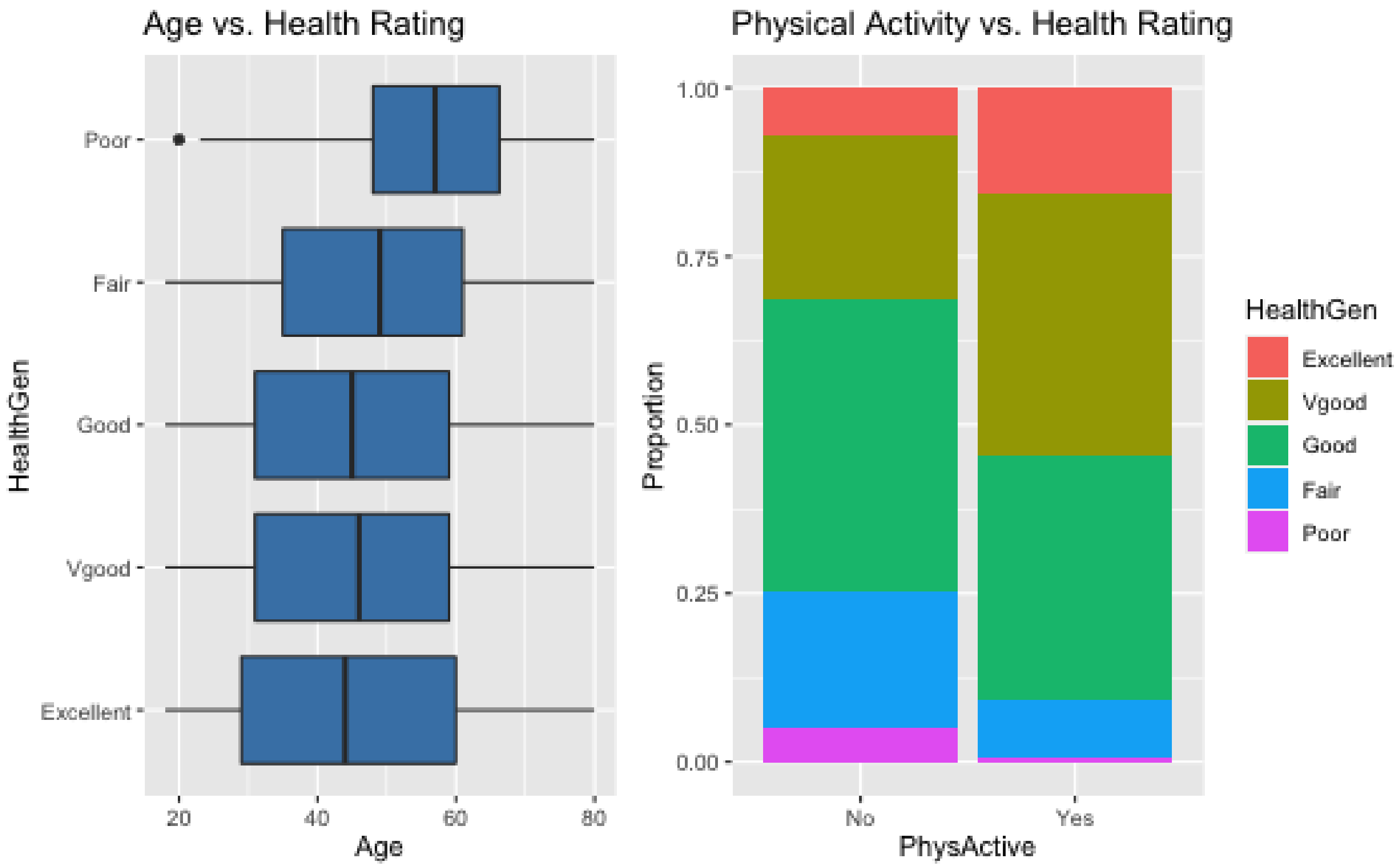
# The data

```
## Rows: 6,710
## Columns: 4
## $ HealthGen  <fct> Good, Good, Good, Good, Vgood, Vgood, Vg
## $ Age        <int> 34, 34, 34, 49, 45, 45, 45, 66, 58, 54,
## $ PhysActive <fct> No, No, No, No, Yes, Yes, Yes, Yes, Yes,
## $ obs_num    <int> 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1
```

# Exploratory data analysis



# Exploratory data analysis



# Model in R

- Use the **`multinom()`** function in the **`nnet`** package

```
library(nnet)
health_m <- multinom(HealthGen ~ Age + PhysActive,
                     data = nhanes_adult)
```

- Put **`results = "hide"`** in the code chunk header to suppress convergence output

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# Output results

```
tidy(health_m, conf.int = TRUE, exponentiate = FALSE) %
  kable(digits = 3, format = "markdown")
```

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# Model output

| y.level | term | estimate | std.error | statistic | p.value | conf.low | conf.high |
|---|---|---|---|---|---|---|---|
| Vgood | (Intercept) | 1.205 | 0.145 | 8.325 | 0.000 | 0.922 | 1.489 |
| Vgood | Age | 0.001 | 0.002 | 0.369 | 0.712 | -0.004 | 0.006 |
| Vgood | PhysActiveYes | -0.321 | 0.093 | -3.454 | 0.001 | -0.503 | -0.139 |
| Good | (Intercept) | 1.948 | 0.141 | 13.844 | 0.000 | 1.672 | 2.223 |
| Good | Age | -0.002 | 0.002 | -0.977 | 0.329 | -0.007 | 0.002 |
| Good | PhysActiveYes | -1.001 | 0.090 | -11.120 | 0.000 | -1.178 | -0.825 |
| Fair | (Intercept) | 0.915 | 0.164 | 5.566 | 0.000 | 0.592 | 1.237 |
| Fair | Age | 0.003 | 0.003 | 1.058 | 0.290 | -0.003 | 0.009 |
| Fair | PhysActiveYes | -1.645 | 0.107 | -15.319 | 0.000 | -1.856 | -1.435 |
| Poor | (Intercept) | -1.521 | 0.290 | -5.238 | 0.000 | -2.090 | -0.952 |
| Poor | Age | 0.022 | 0.005 | 4.522 | 0.000 | 0.013 | 0.032 |

# Fair vs. Excellent Health

The baseline category for the model is `Excellent`.

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The model equation for the log-odds a person rates themselves as having "Fair" health vs. "Excellent" is

$$\log\left(\frac{\hat{\pi}_{Fair}}{\hat{\pi}_{Excellent}}\right) = 0.915 + 0.003 \text{ age} - 1.645 \text{ PhysActive}$$

# Interpretations

$$\log\left(\frac{\hat{\pi}_{Fair}}{\hat{\pi}_{Excellent}}\right) = 0.915 + 0.003 \text{ age} - 1.645 \text{ PhysActive}$$

For each additional year in age, the odds a person rates themselves as having fair health versus excellent health are expected to multiply by 1.003 (exp(0.003)), holding physical activity constant.

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# Interpretations

$$\log\left(\frac{\hat{\pi}_{Fair}}{\hat{\pi}_{Excellent}}\right) = 0.915 + 0.003 \text{ age} - 1.645 \text{ PhysActive}$$

For each additional year in age, the odds a person rates themselves as having fair health versus excellent health are expected to multiply by 1.003 (exp(0.003)), holding physical activity constant.

The odds a person who does physical activity will rate themselves as having fair health versus excellent health are expected to be 0.193 (exp(-1.645 )) times the odds for a person who doesn't do physical activity, holding age constant.

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# Interpretations

$$\log\left(\frac{\hat{\pi}_{Fair}}{\hat{\pi}_{Excellent}}\right) = 0.915 + 0.003 \text{ age} - 1.645 \text{ PhysActive}$$

The odds a 0 year old person who doesn't do physical activity rates themselves as having fair health vs. excellent health are 2.497 (exp(0.915)).

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# Interpretations

$$\log\left(\frac{\hat{\pi}_{Fair}}{\hat{\pi}_{Excellent}}\right) = 0.915 + 0.003 \text{ age} - 1.645 \text{ PhysActive}$$

The odds a 0 year old person who doesn't do physical activity rates themselves as having fair health vs. excellent health are 2.497 (exp(0.915)).

⚠️ **Need to mean-center age for the intercept to have a meaningful interpretation!**

# Hypothesis test for $\beta_{jk}$

The test of significance for the coefficient $\beta_{jk}$ is

**Hypotheses:** $H_0 : \beta_{jk} = 0 \text{ vs } H_a : \beta_{jk} \neq 0$

**Test Statistic:**

$$z = \frac{\hat{\beta}_{jk} - 0}{SE(\hat{\beta}_{jk})}$$

**P-value:** $P(|Z| > |z|)$,

where $Z \sim N(0, 1)$, the Standard Normal distribution

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# Confidence interval for $\beta_{jk}$

- We can calculate the **C% confidence interval** for $\beta_{jk}$ using the following:

$$\hat{\beta}_{jk} \pm z^* SE(\hat{\beta}_{jk})$$

where $z^*$ is calculated from the $N(0, 1)$ distribution

> We are $C\%$ confident that for every one unit change in $x_j$, the odds of $y = k$ versus the baseline will multiply by a factor of $\exp\{\hat{\beta}_{jk} - z^* SE(\hat{\beta}_{jk})\}$ to $\exp\{\hat{\beta}_{jk} + z^* SE(\hat{\beta}_{jk})\}$, holding all else constant.

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# Interpreting confidence intervals for $\beta_{jk}$

| y.level | term | estimate | std.error | statistic | p.value | conf.low | conf.high |
|---|---|---|---|---|---|---|---|
| Fair | (Intercept) | 0.915 | 0.164 | 5.566 | 0.00 | 0.592 | 1.237 |
| Fair | Age | 0.003 | 0.003 | 1.058 | 0.29 | -0.003 | 0.009 |
| Fair | PhysActiveYes | -1.645 | 0.107 | -15.319 | 0.00 | -1.856 | -1.435 |

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| Fair | (Intercept) | 0.915 | 0.164 | 5.566 | 0.00 | 0.592 | 1.237 |
| Fair | Age | 0.003 | 0.003 | 1.058 | 0.29 | -0.003 | 0.009 |
| Fair | PhysActiveYes | -1.645 | 0.107 | -15.319 | 0.00 | -1.856 | -1.435 |

We are 95% confident, that for each additional year in age, the odds a person rates themselves as having fair health versus excellent health will multiply by 0.997 (exp(-0.003)) to 1.009 (exp(0.009)) , holding physical activity constant.

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# Interpreting confidence intervals for $\beta_{jk}$

| y.level | term | estimate | std.error | statistic | p.value | conf.low | conf.high |
|---|---|---|---|---|---|---|---|
| Fair | (Intercept) | 0.915 | 0.164 | 5.566 | 0.00 | 0.592 | 1.237 |
| Fair | Age | 0.003 | 0.003 | 1.058 | 0.29 | -0.003 | 0.009 |
| Fair | PhysActiveYes | -1.645 | 0.107 | -15.319 | 0.00 | -1.856 | -1.435 |

We are 95% confident that the odds a person who does physical activity will rate themselves as having fair health versus excellent health are 0.156 (exp(-1.856 )) to 0.238 (exp(-1.435)) times the odds for a person who doesn't do physical activity, holding age constant.

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# Recap

- Introduce multinomial logistic regression
- Interpret model coefficients
- Inference for a coefficient $\beta_{jk}$